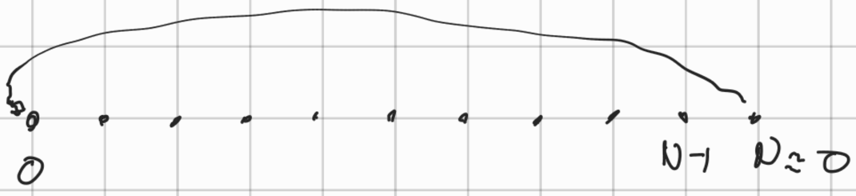
MATH 3235 Probability Theory

11/29/22

Markov Chain.

Random walk



if you are at i at Time t

q To stay at i

$\frac{p}{2} = (1-q)/2$ To jump at $i+1$

$\frac{p}{2}$ To jump at $i-1$

Ex: $q = 0.8$ $\frac{p}{2} = 0.1$

start at 0 at Time 0.

X_t position at Time t

$X_t \in \{0, \ldots, N-1\}$

$t=0 \quad \mathbb{P}(X_0 = 0) = 1$

$t=1 \quad \mathbb{P}(X_1 = 0) = 0.8$

$$\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 0.1$$

$t=2$

$$\begin{aligned}
\mathbb{P}(X_2 = 0) &= \mathbb{P}(X_2 = 0 \mid X_1 = 0)\,\mathbb{P}(X_1 = 0) \\
&\quad + \mathbb{P}(X_2 = 0 \mid X_1 = 1)\,\mathbb{P}(X_1 = 1) \\
&\quad + \mathbb{P}(X_2 = 0 \mid X_1 = -1)\,\mathbb{P}(X_1 = -1) \\
&= 0.64 + 0.01 + 0.01 \\
&= 0.66
\end{aligned}$$

$$\mathbb{P}(X_2 = 2) = 0.01$$

$$\mathbb{P}(X_2 = -2) = 0.01$$

$$\mathbb{P}(X_2 = 1) = \mathbb{P}(X_2 = -1) = 0.16$$

$$\begin{aligned}
\mathbb{P}(X_t = x) &= \mathbb{P}(X_t = x \mid X_{t-1} = x)\,\mathbb{P}(X_{t-1} = x) \\
&\quad + \mathbb{P}(X_t = x \mid X_{t-1} = x-1)\,\mathbb{P}(X_{t-1} = x-1) \\
&\quad + \mathbb{P}(X_t = x \mid X_{t-1} = x+1)\,\mathbb{P}(X_{t-1} = x+1)
\end{aligned}$$

$\mathbb{P}(X_t = x) = p_t(x)$

Trajectories.

$X_0 = 0 \quad X_1 = 1 \quad X_2 = 1$

$0.1 \quad 0.8 \quad = 0.08$

$X_0 = 0 \quad X_1 = 0 \quad X_2 = 0 \quad = 0.64$

Probability to be at 0 at time 2 is the sum of the prob. of all trajectory of 2 step that start at 0 and end at 0.

Suppose we wait a "very long" time will the system "stabilize"?

I start with 10^6 all in 0

$N = 100.$

1) $8 \cdot 10^5$ in 0

$1 \cdot 10^5$ in 1

$1 \cdot 10^5$ in -1

2) $6.4 \; 10^5$ in 0

$1.6 \; 10^5$ in 1 & -1

$0.1 \; 10^5$ in 2 & -2

$\vdots$

After very long time there are roughly 10^4 walker on every site.

$X_t \Rightarrow X$

$$\mathbb{P}(X_t = x) \xrightarrow[t \to \infty]{} p(x)$$

$$\sum_x \left| \mathbb{P}(X_t = x) - p(x) \right| \to 0$$

$$\leq C e^{-\kappa t}$$

$p(x)$ it must be invariant.

$\mathbb{P}(X_t = x) = p(x) \Rightarrow$

$\mathbb{P}(X_{t+1} = x) = p(x)$

$p(x) = \frac{1}{N}$ for every x

$$\mathbb{P}(X_{t+1} = x) = 0.8\, \mathbb{P}(X_{t-1} = x) +$$
$$0.1\, \mathbb{P}(X_{t-1} = x-1) +$$
$$0.1\, \mathbb{P}(X_{t-1} = x+1) =$$
$$0.8\, p(x) + 0.1\, p(x-1) + 0.1\, p(x+1)$$
$$= \frac{1}{N}$$

Markov Chain

Transition matrix

$$T_{xy} = \mathbb{P}(X_t = x \mid X_{t-1} = y)$$

$p_0(x)$ prob of being at x
at Time 0.

Random walk

$$T_{x,x} = q$$

$$T_{x,x-1} = T_{x,x+1} = p/2$$

$$T_{x,y} = 0 \qquad |x-y| > 1$$

$N = 6$

$$\begin{array}{cc}
\begin{pmatrix}
0.8 & 0.1 & 0 & 0 & 0 & 0.1 \\
0.1 & 0.8 & 0.1 & 0 & 0 & 0 \\
0 & 0.1 & 0.8 & 0.1 & 0 & 0 \\
0 & 0 & 0.1 & 0.8 & 0.1 & 0 \\
0 & 0 & 0 & 0.1 & 0.8 & 0.1 \\
0.1 & 0 & 0 & 0 & 0.1 & 0.8
\end{pmatrix} & \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \\
\begin{matrix} 0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 \end{matrix} &
\end{array}$$

-1

$$\mathbb{P}(X_t = x) = \sum_y \mathbb{P}(X_t = x \mid X_{t-1} = y)\,\mathbb{P}(X_{t-1} = y)$$

$$= \sum_y T_{x,y}\,\mathbb{P}(X_{t-1} = y)$$

$$p_t(x) = \mathbb{P}(X_t = x)$$

$$p_t(x) = \sum_y T_{x,y}\,p_{t-1}(y)$$

T is a $N \times N$ matrix

$\vec{p}$ a N vector

$$\vec{p}_t = T\,\vec{p}_{t-1}$$

$T_{x,y} \geq 0$ is a probability

$\sum_x T_{x,y} = 1 \quad \forall y$

Stochastic Matrix

$p_0(x)$ $\quad \sum_x p_0(x) = 1 \quad p(x) \geq 0$

$$p_1(x) = \sum_y T_{x,y}\, p_0(y)$$

$$p_1(x) \geq 0$$

$$\sum_x p_1(x) = \sum_x \sum_y T_{x,y}\, p_0(y) =$$

$$= \sum_y \underbrace{\left(\sum_x T_{x,y}\right)}_{= 1} p_0(y) = \sum_y p_0(y) = 1$$

Steady state.

$$\sum_y T_{xy}\, p(y) = p(x)$$

$$T\vec{p} = \vec{p}$$

$\vec{p}$ is an eigenvector with eigenvalue 1.

$$\sum_x T_{x,y} = 1 \quad \Rightarrow \quad \underline{1}\, T = \underline{1}$$

$$\underline{1} = (1, 1, \ldots 1)$$

$$T^T \underline{1}^T = \underline{1}^T$$

T and T^T have the same eigenvalues!

λ is eigenv. of T^T $\Rightarrow$

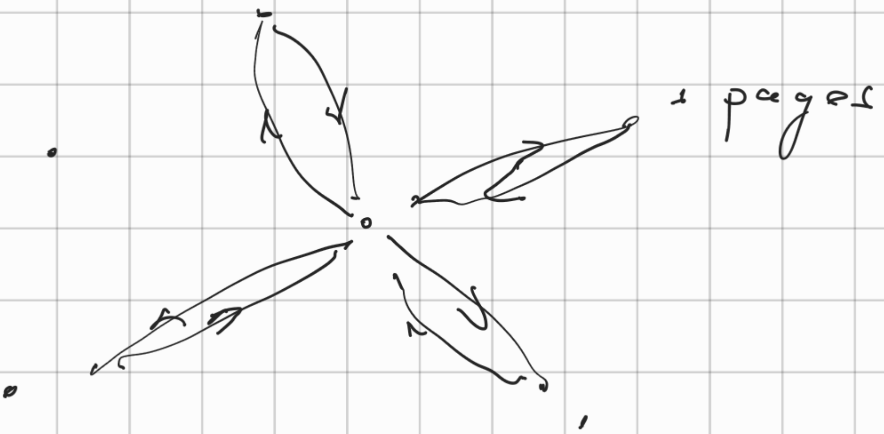
$\det(T^T - \lambda I) = 0 \Rightarrow \det(T - \lambda I) = 0$

$\Rightarrow$ λ is an eigenv. of T.

This implies that every stochast. matrix has a eigenv. 1.

if T is stoch. there exists $\bar{p}$ such that

$$T\bar{p} = \bar{p}$$



$\bar{p}$ is my steady state

$$T_{x,y} = \bar{p}_x = T_{\bar{p}}$$

for any p

$$\sum_y T_{x,y}\, p_y = \bar{p}_x \sum_y p_y = \bar{p}_x$$

Assume That

$$T_{x,y} = S + \mu\, T_{\bar{p}} \qquad S_{xy} \geq 0$$

$$\sum_y T_{x,y}\, \bar{p}_y = \bar{p}_x$$

$$T = S + \mu T_{\bar{p}}$$

$$T p = S p + \mu \bar{p}$$

$$T p = T(S p + \mu \bar{p}) = S^2 p + \mu(1-\mu)\bar{p} + \mu \bar{p}$$

$$T^n p = S^n p + \sum_{i=1}^{n} \mu(1-\mu)^i \bar{p}$$

$$T_{x,y} > 0 \qquad \forall x, y$$

$$\bar{p}_x > 0 \qquad \bar{p}_x = \sum_y T_{xy} \bar{p}_y$$

$$\min_{x,y} T_{x,y} \qquad \max_x \bar{p}_x$$

I can find μ such That

$$T_{x,y} - \mu p_x > 0 \qquad \forall_{x,y}$$